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Projective Solution of Integral Equations Arising in Electric and Magnetic Field Problems

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Many integral equations arising in electric and magnetic field problems have Green's function kernels, which cause difficulty in integration because of their essential singularities. By performing a simple coordinate rotation, these singularities can be shifted entirely onto one coordinate axis, so that iterated weighted numerical quadratures may be applied to evaluate exactly the projective representation of such integral operators on suitable finite-dimensional function spaces. Standard projective methods may therefore be applied to solve such integral equations accurately with very little computational effort.

INTRODUCTION

A large number of boundary-value problems arising in electric and magnetic field calculations may be treated best by formulation in terms of integral equations. For example, various electrostatics problems arising in connection with transmission lines [1, 2], skin effect calculations for nonmagnetic conductors [3] and many others that involve unbounded solution regions, have been dealt with by expressing the fields in Fredholm integral equations of the first or second kinds. The major advantages to be realized in such treatment are (1) the inclusion of all boundary conditions in the integral operator, and (2) in many problems, reduction of the number of space variables by one. Especially for problems naturally posed in two dimensions, the latter advantage may be of very great importance.

Integral equations are usually very difficult to solve analytically, so that numerical methods are normally resorted to. A major difficulty, however, obtrudes in th fact that integral equations arising in electric or magnetic field theory have singula kernels, while the approximation theory on which most numerical methods rest presupposes continuous kernels. This fact has often hindered the development of high-accuracy numerical techniques, and has led to the use of a variety of ingenious special approximations for particular problems [4, 5]. The object of the present paper is to show that, by correct choice of the order of integration, the singularities

can often be taken into account exactly, so that computationally superior methods can be developed readily. As an illustration of this technique, a method is given for determining the electrostatic charge distribution on an infinitely long cylindrical conductor of arbitrary polygonal cross-sectional shape, a problem intimately related to a wide variety of transmission-line parameter calculations.

Formulation of the Integral Equations

It is usual to regard electromagnetic field problems as being composed of a differential equation

$$\mathscr{D}\phi = g \tag{1}$$

subject to some set of boundary constraints

$$\mathscr{B}\phi = 0, \tag{2}$$

where g is a source distribution, and ϕ is the desired field. The differential operators $\mathscr{D}$ and $\mathscr{B}$ may be scalar, vector, or dyadic, depending on the nature of the variables ϕ and g. The equivalent integral equation formulation is obtained by first solving

$$\mathscr{D}\Phi = \delta(x), \tag{3}$$

where $\delta(x)$ is an impulse source function (of appropriate dimensionality) located at point x, subject to the homogeneous boundary requirements of Eq. (2). The result at a point y will be denoted by $\Phi(x; y)$, and is termed the *fundamental solution.* A solution to the original problem is recoverable by superposition of fundamental solutions, in the form

$$\phi(y) = \int g(x)\, \Phi(x; y)\, dx. \tag{4}$$

To furnish a simple example, the distribution of electric scalar potential in the space outside an infinitely long charged cylindrical conductor, as in Fig. 1, is given by Poisson's equation

$$\nabla^2\phi = -\rho/\epsilon_0 \tag{5}$$

whose fundamental solution is

$$\Phi(x; y) = \frac{1}{2\pi} \log r_{xy}\,. \tag{6}$$

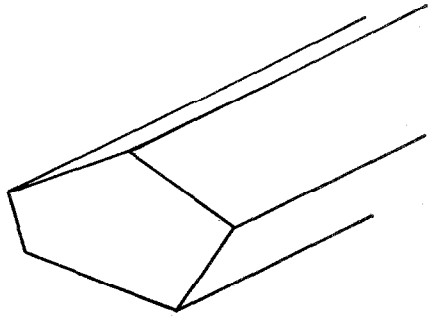

FIG. 1. Infinitely long electrically charged cylindrical conductor.

Here r_{xy} denotes the distance between points x and y. Thus

$$\phi(y) = \int -\frac{\log r_{xy}}{2\pi\epsilon_0}\,\rho(x)\,dx. \tag{7}$$

More generally, in problems involving inhomogeneous media or other added constraints, the fundamental solution may assume a more complicated form. Nevertheless, it is usually characterized by a factorable singularity at $x = y$ (most often logarithmic in two-dimensional problems, and of form r_{xy}^{-1} in three dimensions). That is, fundamental solutions of two-dimensional problems may be written in the form

$$\Phi(x; y) = \left[\frac{\Phi(x; y)}{\log r_{xy}}\right] \log r_{xy}\,, \tag{8}$$

where the bracketed factor on the right is a continuous function. Even in cases in which the singularity has other than logarithmic or inverse-square behavior, it is commonly possible to write

$$\Phi(x; y) = \left[\frac{\Phi(x; y)}{s(r_{xy})}\right] s(r_{xy}), \tag{9}$$

where again the bracketed factor is continuous, and $s(r_{xy})$ is singular at $r_{xy} = 0$. An example of a very complicated fundamental solution, whose essentially logarithmic singularity is nevertheless readily identified, will be found in Ref. [1].

Stated in fully general form, therefore, the problem to be considered is to solve an integral equation containing a singularity function in the kernel:

$$\phi(y) = \int c(x; y)\, s(r_{xy})\, g(x)\, dx. \tag{10}$$

It is assumed that $c(x; y)$ is a continuous function, while $s(r_{xy}) = s(|\, x - y\,|)$

possesses a singularity at $x = y$; the integration is taken to encompass some finite region of space.

Solution by Projection Methods

In general, the solution of linear operator equations of the form

$$\phi = \mathscr{L} g, \tag{11}$$

where $\mathscr{L}$ is some linear operator, may be carried out by the following numerical approximation procedure [6, 7]. Let $\{\alpha_i \mid i = 1,..., N\}$ denote a finite set of linearly independent functions by which g may be approximated,

$$g = \sum a_i \alpha_i . \tag{12}$$

Let $\{\beta_j \mid j = 1,..., M\}$ denote another finite set of linearly independent functions. By forming inner products (according to some convenient definition) of (11) with each β_j in turn, M simultaneous linear equations are generated:

$$\langle \phi, \beta_j \rangle = \sum_i \langle \mathscr{L} \alpha_i , \beta_j \rangle a_i . \tag{13}$$

Since each projection on the left is simply a number, (13) is usually written as a matrix equation

$$f = La, \tag{14}$$

where

$$f_k = \langle \phi, \beta_k \rangle, \tag{15}$$

$$L_{jk} = \langle \mathscr{L} \alpha_j , \beta_k \rangle. \tag{16}$$

In order to keep the method computationally simple, it is quite common to select function sets so that $N = M$, resulting in an invertible square matrix L. It is also a frequent practice to employ the same set of functions in both roles (the Galerkin method).

A quite worrisome difficulty in the practical application of this method to integral operators lies in the fact that the kernel functions contain the singular function $s(x; y)$. If one takes as the definition of an inner product the extremely common form [8]

$$\langle a, b \rangle \quad \int a^* b \, dS, \tag{17}$$

considerable difficulty can be encountered in evaluating the elements of L, since they will be given by

$$L_{jk} = \iint s(|\, x - y \,|)\, c(x; y)\, \alpha_j(y)\, \alpha_k(x)\, dx\, dy. \tag{18}$$

Unless the integrand contains particularly simple functions, analytic integration is fraught with difficulties. In consequence, the function sets used in most problems solved by projection methods have been relatively simple.

Evaluation of Singular Iterated Integrals

In numerous two-dimensional boundary-value problems, the integral given by (18) represents an iterated contour integral.

In this case, it is possible to devise a rigorously valid numerical integration method which fully accounts for the singularity of $s(|\, x - y \,|)$.

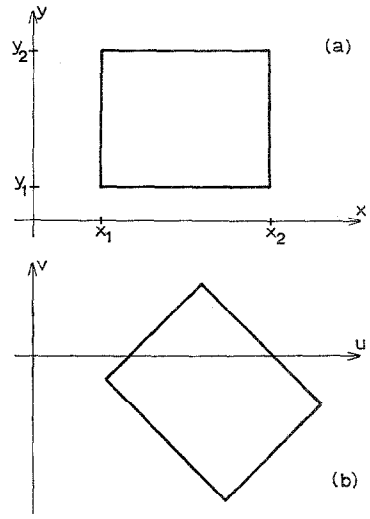

FIG. 2. Region of integration for Eq. (18): (a) as represented in $x - y$ plane; (b) as represented in $u - v$ plane.

In the $x - y$ plane, Eq. (18) represents a surface integral over a rectangular region, as indicated in Fig. 2*a*. The integrand is continuous everywhere in this plane, except along the line $x = y$, where it is singular .Let now the coordinate transformation be made

$$u = \tfrac{1}{2}(x + y), \tag{19}$$

$$v = \tfrac{1}{2}(-x + y), \tag{20}$$

thereby transforming the region of integration into the region in Fig. 2b. The integral to be evaluated, Eq. (18), changes accordingly into

$$L_{jk} = 2\int_{v_1}^{v_2} s(|-2v|)\int_{u_1}^{u_2} c(u-v;\, u+v)\,\alpha_k(u+v)\,\alpha_k(u-v)\,du\,dv, \tag{21}$$

where the limits of integration, as is obvious from Fig. 2b, are somewhat complicated. For the integration with respect to u, either of two distinct lower limits will apply, depending on the value of v:

$$\begin{aligned} u_1 &= x_1 + v, \qquad \frac{y_1 - x_1}{2} \leqslant v \leqslant \frac{y_2 - x_1}{2}, \\ &= y_1 - v, \qquad \frac{y_1 - x_2}{2} \leqslant v \leqslant \frac{y_1 - x_1}{2}. \end{aligned} \tag{22}$$

Similarly, the upper limits are given by

$$\begin{aligned} u_2 &= x_2 + v, \qquad \frac{y_1 - x_2}{2} \leqslant v \leqslant \frac{y_2 - x_2}{2}, \\ &= y_2 - v, \qquad \frac{y_2 - x_2}{2} \leqslant v \leqslant \frac{y_2 - x_1}{2}. \end{aligned} \tag{23}$$

Since all the functions α_i, as well as c, are continuous, there is no fundamental difficulty in constructing approximations to the function

$$w(v) = \int_{u_1}^{u_2} c\alpha_j\alpha_i\,du \tag{24}$$

by numerical quadratures, for any given value of v; both the integrand and the limits of integration are well defined. The formulation of L_{jk}, as in (21), may then be accomplished by finding

$$L_{jk} = 2\int_{v_1}^{v_2} s(|-2v|)\,w(v)\,dv. \tag{25}$$

However, this integration presents no problem, provided s is a function such that there exist weighted quadrature formulas of the type

$$\int_0^1 s(z)f(z)\,dz = \sum_i d_i f(z_i) \tag{26}$$

for in such a case, L_{jk} may be written as

$$L_{jk} = 2\int_0^{v_2} s(|-2v|)\,w(v)\,dv - 2\int_0^{v_1} s(|-2v|)\,w(v)\,dv \tag{27}$$

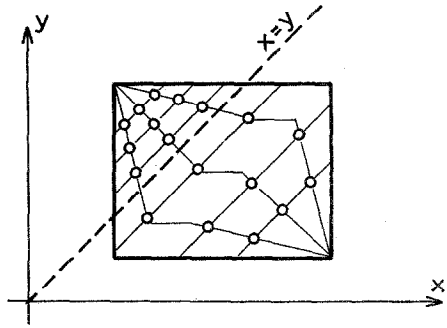

FIG. 3. Locations of three-point Gaussian product quadrature nodes, in $x - y$ plane, for evaluation of Eqs. (18). The nodes shown will yield exact integration for a biquintic integrand.

and the quadrature formula (26) applied twice. It remains to be noted that for the most common types of singularity arising in electric or magnetic field problems, e.g., $s(z) = \log z$, suitable quadrature formulas either have already been tabulated [9], or can be derived by following established methods. The combination of numerical quadratures applied to (25) and (24) in turn, yields integration methods that closely resemble those well established for cones [10], except for the weighted nature of one of the formulas. For example, if three-point Gaussian formulas are used in u and v in turn, the array of integration nodes for the region shown in Fig. 2a appears as in Fig. 3. Naturally it is only necessary to determine the node locations and quadrature weights using the argument indicated above; there is no need to carry out the actual coordinate transformations in the computer programming.

APPLICATIONS TO ELECTROSTATICS PROBLEMS

The above method of forming projections can be applied to any set of functions $\{\alpha\}$ whose integrals can be well approximated by Gaussian (or other) quadrature formulas. Since an n-point Gaussian rule is known to integrate a polynomial of degree $2n - 1$ exactly, polynomials readily suggest themselves as suitable expansion functions. Because any complete set of linearly independent polynomials of given degree spans the same function space, there is no particular reason for choosing any one kind of polynomial over another, except computational convenience. For the present examples, interpolation polynomials of closed type, with equispaced nodes (the Newton–Cotes interpolation polynomials) were used; as an alternative, the monomials 1, x, x^2,... were also tried. It is gratifying to report that the results from both sets were identical, within the arithmetic error limits of the machine used.

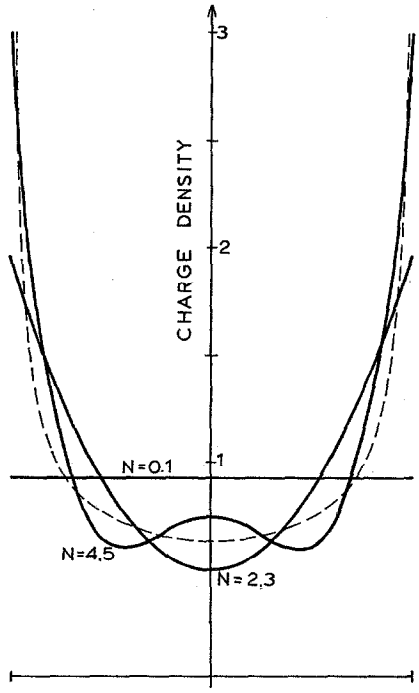

FIG. 4. Relative charge density distribution on an infinitely long, very thin, flat, conductive ribbon. The broken line indicates the exact solution; N is the degree of the polynomial approxima tion employed.

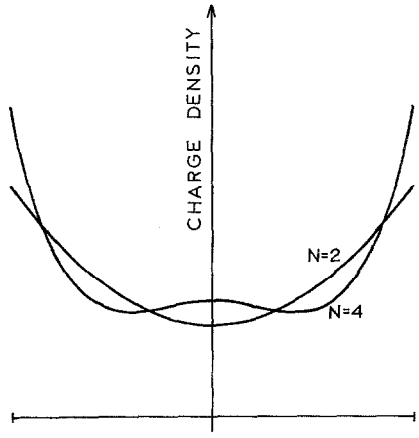

FIG. 5. Relative charge density distribution on one side of a square, infinitely long, conductor, calculated using two different degrees N of polynomial approximation.

Figure 4 exhibits the results obtained for the classical, and analytically solvable, problem of charge distribution on a thin flat strip conductor. The analytic solution is also shown for comparison. It is immediately apparent that the approximate solutions exhibit the oscillatory behaviour about the true solution that is normally expected from least-squares approximations. Since the solution must be an even function, no difference should be observed between approximations with polynomials of degree $2n$, or $2n + 1$; this is indeed the case. Solutions appear for polynomial sets up to quadratic and quartic, i.e., the sets $\{1, x^2\}$ and $\{1, x^2, x^4\}$, respectively. Despite the very small function sets employed, the total charge on the strip is found to better than 1 % by this means.

In Figure 5, corresponding results are exhibited for a square charged conductor. Here it is convenient to choose trial functions which have polynomial behavior along one side of the square, and are held at zero on the remaining sides; the integrations are thereby simplified as compared to functions possessing nonzero values everywhere. Again, the solutions exhibit an oscillatory behavior.

It ought to be noted that the accuracies obtained by the method described in this paper are comparable with those of the solution in [1]; however, the matrix problem to be solved in the latter case had a dimensionality five to fifteen times larger. Consequently, substantially longer computing times were required, even though the individual matrix elements could be evaluated with somewhat less work.

Conclusions

The method described in this paper for forming projective approximations of integral operators has wide application in numerous continuum problems, in fact wherever Green's functions appear as kernels of the integral operator. In applications to electrostatics problems, the ability to form accurate projections onto essentially arbitrary function sets has been found to provide fast computing times, and it is anticipated that similar results will hold true for integral operators of greater complexity.

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